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On Optimal Tensor Product Approximation

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INTRODUCTION

In connection with the finite element method, tensor product schemes of interpolation have widely been used [3, 6, 13]. It is the purpose of this paper to treat these interpolation methods in the abstract setting of the theory of optimal approximation developed by Sard [10–12]. This approach enables us to compute explicit expressions for the norms of certain error functionals.

1. OPTIMAL INTERPOLATION

The theory of optimal interpolation as a special case of the theory of optimal approximation in the sense of Sard [10–12] is characterized by the tuple

$$(X, Y, Z; U, F) \tag{1.1}$$

Here X, Y, Z are (separable, complex) Hilbert spaces and

$$U : X \to Y, \qquad F : X \to Z$$

are continuous linear mappings. Let us assume that the completeness condition holds [11]. Thus the Hermitian form

$$((x, y)) = (Ux, Uy) + (Fx, Fy) \tag{1.2}$$

($x, y \in X$) defines a scalar product on X, which induces the original topology on X.

We define the orthogonal projector P in the space $(X, ((\cdot, \cdot)))$ by Ker $P =$ Ker F. P is called the *spline projector* corresponding to (1.1). The following theorem shows that the *problem of optimal interpolation* can be solved by P.

THEOREM 1.1 (Sard [11]). *For any $x \in X$ the element $\xi = Px$ is the unique element among all $y \in X$ satisfying $Fy = Fx$, which minimizes the functional*

$$y \to \| Uy \| .$$

Remark 1.2. If

$$\operatorname{Ker} U = \operatorname{Ker} F^{\perp}, \tag{1.3}$$

the tuple $(X, Y, Z; U, F)$ describes a *problem of standard interpolation*. In this case we have $F\xi = Fx$ and $U\xi = 0$. Because of (1.3), the sets $\operatorname{Im} U$ and $\operatorname{Im} F$ are complete. Accordingly, in the case of standard interpolation we may and shall assume that $\operatorname{Im} U = Y$ and $\operatorname{Im} F = Z$. Now by Banach's theorem there exists a unique continuous linear mapping $T_F : Y \to X$ satisfying $T_F U = \mathrm{id}_X - P$ and $UT_F = \mathrm{id}_Y$, where id_X, id_Y denote the identities on X, and Y, respectively. In this context we can consider the decomposition

$$x = Px + T_F(Ux) \tag{1.4}$$

($x \in X$) as the *generalized Taylor formula* of the problem of *standard interpolation* $(X, Y, Z; U, F)$.

If not mentioned otherwise, let X in the sequel be equipped with scalar product (1.2).

THEOREM 1.3. *Let $G : X \to W$ be a continuous, surjective, linear mapping from X to the (separable, complex) Hilbert space W which satisfies*

$$\operatorname{Ker} G \subset \operatorname{Ker} F. \tag{1.5}$$

Then the tuple

$$(X, Y, W; U, G) \tag{1.6}$$

defines an optimal interpolation problem *and the orthogonal projector Q in $(X, ((\cdot, \cdot)))$ defined by*

$$\operatorname{Ker} Q = \operatorname{Ker} G \tag{1.7}$$

is the spline projector for problem (1.6).

Proof. By (1.7), the equations

$$\operatorname{Ker} Q = \operatorname{Im}(\mathrm{id}_X - Q) = \operatorname{Ker} G$$

hold, and thus the equation

$$GQ = G \tag{1.8}$$

is valid too. Furthermore, Banach's theorem guarantees the existence of a continuous linear mapping $G^{-1} : W \to X$ satisfying

$$G^{-1}G = Q. \tag{1.9}$$

From these premises we infer the existence of a constant $c > 0$ such that

$$\| Gx \| \geqslant c \cdot \| Qx \| \geqslant c \cdot \| Px \|$$

($x \in X$). The second inequality is an immediate consequence of $\operatorname{Im} Q \supset \operatorname{Im} P$, whereas the first follows by (1.9).

Now we have

$$\begin{aligned} \| Gx \|^2 + \| Ux \|^2 &\geqslant \min(c, 1) \cdot \{\| Px \|^2 + \| Ux \|^2\} \\ &= \min(c, 1) \cdot \| x \|^2. \end{aligned}$$

Thus problem (1.6) defines an optimal interpolation problem in the sense of Sard and we have established the first part of our statement. Because of (1.8) we merely have to verify that the projector Q defined by (1.7) is an orthogonal projector relative to the scalar product

$$((x, y))' = (Ux, Uy) + (Gx, Gy). \tag{1.10}$$

Now relation (1.5) yields

$$FQy = Fy$$

and

$$(UQy, U(y - Qy)) = (Qy, y - Qy) = 0.$$

Hence

$$((Qy, y - Qy))' = 0.$$

Thus Q is an orthogonal projector relative to (1.10).

2. Optimal Tensor Product Interpolation

This section is concerned, first, with the description of the simplest case of optimal interpolation in $X_1 \otimes X_2$, the standard tensor product interpolation. Among other topics, a suitable Hilbert space structure on $X_1 \otimes X_2$ is discussed. (For tensor products of Hilbert spaces and linear mappings we refer to [1, pp. 39–50].)

Let us start with two standard interpolation problems

$$(X_j, Y_j, Z_j; U_j, F_j) \qquad (j = 1, 2). \tag{2.1}$$

In particular,

$$\operatorname{Ker} U_j = \operatorname{Ker} F_j^{\perp} \qquad (j = 1, 2).$$

Since X_1 and X_2 are equipped with the scalar product induced by (2.1) the scalar product on $X_1 \otimes X_2$ is of the form

$$\begin{aligned}(x, y) = {} & (F_1 \otimes F_2(x), F_1 \otimes F_2(y)) \\ & + (F_1 \otimes U_2(x), F_1 \otimes U_2(y)) \\ & + (U_1 \otimes F_2(x), U_1 \otimes F_2(y)) \\ & + (U_1 \otimes U_2(x), U_1 \otimes U_2(y))\end{aligned} \tag{2.2}$$

($x, y \in X_1 \otimes X_2$). Put [5]

$$V = F_1 \otimes U_2 \times U_1 \otimes F_2 \times U_1 \otimes U_2 .$$

The problem of standard tensor product interpolation is characterized by the tuple

$$(X_1 \otimes X_2 , Z_1 \otimes Y_2 \times Y_1 \otimes Z_2 \times Y_1 \otimes Y_2 , Z_1 \otimes Z_2 ; V, F_1 \otimes F_2). \tag{2.3}$$

It is immediate that the completeness condition holds for (2.3). Let P_j ($j = 1, 2$) denote the spline-projectors corresponding to (2.1). This leads us to

THEOREM 2.1. *Let $x \in X_1 \otimes X_2$. Then $\xi = P_1 \otimes P_2(x)$ is the unique element of $X_1 \otimes X_2$ satisfying the relations*

$$F_1 \otimes F_2(\xi) = F_1 \otimes F_2(x), \qquad V(\xi) = 0.$$

Proof. As is well known, $P_1 \otimes P_2$ is an orthogonal projector on $X_1 \otimes X_2$ (equipped with scalar product (2.2)) which satisfies

$$(F_1 \otimes F_2)(P_1 \otimes P_2) = F_1 \otimes F_2 ,$$

and

$$V(P_1 \otimes P_2) = 0.$$

This proves the theorem. (Note that [5, pp. 62–67]

$$\operatorname{Im} P_1 \otimes P_2 = \operatorname{Ker} F_1 \otimes F_2^{\perp} = \operatorname{Ker} V.)$$

In the present case of standard interpolation we may and shall assume that

$$\operatorname{Im} U_j = Y_j \qquad (j = 1, 2), \tag{2.4}$$

$$\operatorname{Im} F_j = Z_j \qquad (j = 1, 2). \tag{2.5}$$

An immediate consequence is $\operatorname{Im} F_1 \otimes F_2 = Z_1 \otimes Z_2$, and we are able to prove the following theorem.

THEOREM 2.2. *Under assumptions* (2.4) *and* (2.5),

$$\operatorname{Im} V = Z_1 \otimes Y_2 \times Y_1 \otimes Z_2 \times Y_1 \otimes Y_2 .$$

Proof. Because of (2.4) and (2.5) there exist continuous linear mappings

$$T_{F_j}: Y_j \to X_j \qquad (j = 1, 2),$$

$$T_{U_j}: Z_j \to X_j \qquad (j = 1, 2),$$

which satisfy

$$T_{F_j} U_j = \mathrm{id}_{X_j} - P_j \qquad (j = 1, 2),$$

$$T_{U_j} F_j = P_j \qquad (j = 1, 2).$$

By defining the continuous linear mapping

$$T_{F_1 \otimes F_2}: Z_1 \otimes Y_2 \times Y_1 \otimes Z_2 \times Y_1 \otimes Y_2 \to X_1 \otimes X_2 ,$$

$$T_{F_1 \otimes F_2}(x_{01}, x_{10}, x_{11}) = T_{U_1} \otimes T_{F_2}(x_{01}) + T_{F_1} \otimes T_{U_2}(x_{10}) + T_{F_1} \otimes T_{F_2}(x_{11})$$

$$V T_{F_1 \otimes F_2} = \mathrm{id}_{Z_1 \otimes Y_2 \times Y_1 \otimes Z_2 \times Y_1 \otimes Y_2} ,$$

$$T_{F_1 \otimes F_2} V = \mathrm{id}_{X_1 \otimes X_2} - P_1 \otimes P_2 .$$

The theorem follows from the first statement.

The second statement implies a decomposition similar to (1.4),

$$x = P_1 \otimes P_2(x) + T_{F_1 \otimes F_2}(Vx)$$

($x \in X_1 \otimes X_2$), a relation which can be considered as the *generalized Taylor formula* of the problem of *standard tensor product interpolation* (2.3).

THEOREM 2.3. *Let*

$$G_j : X_j \to W_j \qquad (j = 1, 2)$$

be continuous linear mappings from X_j ($j = 1, 2$) *onto the* (*separable, complex*) *Hilbert spaces* W_j ($j = 1, 2$) *satisfying*

$$\operatorname{Ker} G_j \subset \operatorname{Ker} F_j \qquad (j = 1, 2).$$

Then the tuple

$$(X_1 \otimes X_2 , Z_1 \otimes Y_2 \times Y_1 \otimes Z_2 \times Y_1 \otimes Y_2 , W_1 \otimes W_2 ; V, G_1 \otimes G_2) \quad (2.6)$$

defines a problem of optimal tensor product interpolation. *Furthermore, let* Q_j ($j = 1, 2$) *be the orthogonal projectors of* X_j ($j = 1, 2$) *with inner products* (1.2) *defined by*

$$\operatorname{Ker} Q_j = \operatorname{Ker} G_j \qquad (j = 1, 2).$$

Then the spline projector corresponding to (2.6) *is given by* $Q = Q_1 \otimes Q_2$.

Proof. As in the proof of Theorem 1.3, there exist continuous linear mappings

$$G_j^{-1}: W_j \to X_j \qquad (j = 1, 2)$$

which satisfy

$$G_j^{-1} G_j = Q_j \qquad (j = 1, 2).$$

We deduce the relations

$$(G_1^{-1} \otimes G_2^{-1})(G_1 \otimes G_2) = Q_1 \otimes Q_2,$$
$$(G_1 \otimes G_2)(Q_1 \otimes Q_2) = G_1 \otimes G_2,$$

which imply

$$\text{Ker } Q_1 \otimes Q_2 = \text{Ker } G_1 \otimes G_2,$$
$$\text{Im } Q_1 \otimes Q_2 = \text{Ker } G_1 \otimes G_2{}^{\perp}.$$

In view of Theorem 1.3, we now merely have to prove

$$\text{Ker } G_1 \otimes G_2 \subset \text{Ker } F_1 \otimes F_2 .$$

For every $x \in X_1 \otimes X_2$ with

$$G_1 \otimes G_2(x) = 0$$

we have

$$Q_1 \otimes Q_2(x) = 0.$$

As

$$(F_1 \otimes F_2)(Q_1 \otimes Q_2) = F_1 \otimes F_2,$$

it follows that

$$\text{Ker } G_1 \otimes G_2 \subset \text{Ker } F_1 \otimes F_2 .$$

By Theorem 1.3, the completeness condition for (2.6) holds.

3. Optimal Approximation on Linear Functionals

We will now study the problem of optimal approximation of linear functionals. Therefore we assume $L \in X^*$. As in [11] we define the class of admissible approximations of L for the problem $(X, Y, W; U, G)$ as

$$\mathfrak{A}(L) = \{EG : E \in W^*, \text{Ker } U \subset \text{Ker}(L - EG)\}.$$

Because $LQ = LG^{-1}G \in \mathfrak{A}(L)$, $\mathfrak{A}(L)$ is not empty.

LEMMA 3.1. *For every $EG \in \mathcal{O}(L)$ we have*

$$L - EG = K_E U \tag{3.1}$$

with $K_E \in Y^*$ satisfying

$$\| L - EG \| = \| K_E \| . \tag{3.2}$$

Proof. For (3.1) we refer to [11]. Let λ denote the representer (dual) of $L - EG$: $(L - EG)(x) = (x, \lambda)$. Since Ker $U = \text{Im } P \subset \text{Ker}(L - EG)$, we obtain $P\lambda = 0$. Taking into account Ker P = Ker F, this implies $F\lambda = 0$, and finally, $\| L - EG \| = \| U\lambda \|$. On the other hand we have

$$(L - EG)(x) = (Ux, U\lambda) = K_E(Ux).$$

The relation Im $U = Y$ implies

$$\| K_E \| = \| U\lambda \| = \| L - EG \| .$$

This proves (3.2).

Because of (3.1) we have

$$| L(x) - EG(x)| \leqslant \| K_E \| \cdot \| Ux \|$$

$(x \in X)$.

This inequality motivates the following definition of optimal approximation of L. The functional $E_0 G \in \mathcal{O}(L)$ is called an *optimal approximation* of L (with respect to G and U) iff $\| K_{E_0} \| \leqslant \| K_E \|$ $(EG \in \mathcal{O}(L))$. Taking into account (3.2) this is equivalent to

$$\| L - E_0 G \| = \min_{EG \in \mathcal{O}(L)} \| L - EG \|.$$

The following theorem shows how to calculate the optimal approximation.

THEOREM 3.1 (Sard [11]). *The optimal approximation of L is given by $E_0 G = LQ$ $(E_0 = LG^{-1})$.*

We now consider the optimal approximation of linear functionals of the special form $L = L_1 \otimes L_2$, with $L_j \in X_j^*$ $(j = 1, 2)$ for problem (2.6). An application of Theorems 3.1 and 2.3 yields immediately the following result.

THEOREM 3.2. *The optimal approximation of $L_1 \otimes L_2$ (with respect to $G_1 \otimes G_2$ and V) is given by*

$$E_0(G_1 \otimes G_2) = L_1 Q_1 \otimes L_2 Q_2 ,$$

with

$$E_0 = L_1 G_1^{-1} \otimes L_2 G_2^{-1}.$$

Our final purpose is to derive an explicit expression for the norm

$$\| L_1 \otimes L_2 - L_1Q_1 \otimes L_2Q_2 \| .$$

THEOREM 3.3. *The norm of the remainder functional* $L_1 \otimes L_2 - L_1Q_1 \otimes L_2Q_2$ *is given by*

$$\begin{aligned} &\| L_1 \otimes L_2 - L_1Q_1 \otimes L_2Q_2 \|^2 \\ &\quad = \| L_1 - L_1Q_1 \|^2 \cdot \| L_2Q_2 \|^2 + \| L_1Q_1 \|^2 \cdot \| L_2 - L_2Q_2 \|^2 \\ &\quad + \| L_1 - L_1Q_1 \|^2 \cdot \| L_2 - L_2Q_2 \|^2 . \end{aligned}$$

Proof. The proof uses the following

LEMMA 3.2. *Let* $L \in X^*$ *and* A, B *be orthogonal projectors on* X, *which satisfy either one (and hence both) of the orthogonal relations* $AB = BA = 0$. *Then*

$$\| LA + LB \|^2 = \| LA \|^2 + \| LB \|^2.$$

Proof. In general we have $(LA + LB)(x) = (x, A\eta) + (x, B\eta)$ with η the representer (dual) of L. Since $(A\eta, B\eta) = 0$,

$$\| A\eta + B\eta \|^2 = \| A\eta \|^2 + \| B\eta \|^2.$$

This implies the lemma.

If we now consider the decomposition

$$\begin{aligned} L_1 \otimes L_2 - L_1Q_1 \otimes L_2Q_2 = L_1 \otimes L_2\{(\mathrm{id}_{X_1} - Q_1) \otimes Q_2 + Q_1 \otimes (\mathrm{id}_{X_2} - Q_2) \\ + (\mathrm{id}_{X_1} - Q_1) \otimes (\mathrm{id}_{X_2} - Q_2)\}, \end{aligned}$$

our theorem is verified by the validity of the orthogonal relations

$$\begin{aligned} ((\mathrm{id}_{X_1} - Q_1) \otimes Q_2)(Q_1 \otimes (\mathrm{id}_{X_2} - Q_2)) &= 0, \\ ((\mathrm{id}_{X_1} - Q_1) \otimes Q_2)((\mathrm{id}_{X_1} - Q_1) \otimes (\mathrm{id}_{X_2} - Q_2)) &= 0, \\ (Q_1 \otimes (\mathrm{id}_{X_2} - Q_2))((\mathrm{id}_{X_1} - Q_1) \otimes (\mathrm{id}_{X_2} - Q_2)) &= 0, \end{aligned}$$

and an application of Lemma 3.2 and Theorem 3.1.

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